ON A THEOREM OF CORSON AND LINDENSTRAUSS ON LINDELÖF FUNCTION SPACES

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ABSTRACT

We give two generalizations of a theorem of H. H. Corson and J. Lindenstrauss concerning Lindelöf property in spaces of functions vanishing at infinity.

The aim of this paper is to prove the following two theorems (terminology and notations are explained below):

THEOREM 1. *If X is pseudometrizable, locally separable and Y an $\aleph_0$-space equipped with its fine uniformity, then every subset A of $C_0(X, Y)$, invariant under projections, is a Lindelöf space for the topology of uniform convergence on members of any Morita decomposition of X.*

THEOREM 2. *If X is any pseudometrizable space and Y an $\aleph_0$-space, then every subspace A of $C_0(X, Y)$, invariant under projections, is a Lindelöf space for the topology of pointwise convergence.*

which generalize [2, 2.5]. Theorem 1 implies that a subset of $C_0(\Gamma)$, Γ a discrete space, invariant under projections is a Lindelöf space for the topology of uniform convergence on disjoint countable subsets of Γ. It improves [1, Proposition 3] by showing that the σ-product of $\aleph_0$-spaces is a Lindelöf space for a topology finer than the product topology. We note that the difference between Theorem 2 and Theorem 1 is not so large as it appears formally. For, a theorem of J. R. Isbell, [3, 8.4], asserts that $X \setminus f^{-1}(0)$ is locally compact for $f \in C_0(X, Y)$. Therefore $C_0(X, Y)$ is reduced to a point whenever no point of X has a compact neighborhood. The difficulty concerning general X's is in working with the

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topology of $C_0(X, Y)$ as well as with the restrictions of functions in $C_0(X, Y)$ (see the proof of Theorem 2). Now we explain our terminology and notations:

NOTATIONS. We shall denote by

$C_0(X, Y)$ the set of all mappings $X \to Y$ having as limit at infinity a given point $0 \in Y$.

$w(X)$ the weight of the topological space X = minimum cardinal of a basis of the topology of X.

$dc(X)$ the density character of the topological space X = minimum cardinal of a dense subset.

$bc(X)$ the basis character of the uniform space X = minimum cardinal of a basis of the uniformity of X.

$W(K_1, \cdots, K_n; U_1, \cdots, U_n)$ the set of all mappings f in a given set A such that $f(K_i) \subseteq U_i$ for all $i = 1, \cdots, n$.

$A|_E$ the set $\{f|_E \mid f \in A\}$ for any set A of mappings.

TERMINOLOGY. $A \subseteq C_0(X, Y)$ *is invariant under projections* means that for $f \in A$ and $U \subseteq X$ clopen (= closed and open) and separable, A contains the map which agrees with f on U and is 0 on $X \setminus U$. By *Morita decomposition* of a locally separable, pseudometrizable space X we understand a decomposition of X made of separable, disjoint, clopen subsets (whose existence is given by a well known theorem of K. Morita). According to [4], an $\aleph_0$-*space* is a regular space X with a countable $\mathscr{P} \subseteq \mathscr{P}(X)$, called *pseudobase*, such that for every compact $K \subseteq X$ and every open $U \subseteq X$ containing K, there is $P \in \mathscr{P}$ such that $K \subseteq P \subseteq U$. An $\aleph_0$-space is completely regular, hence uniformizable, since it is Lindelöf. The *fine uniformity* is the finest compatible uniformity.

The proof of Theorem 2 is based on Theorem 1, while the proof of Theorem 1 is based on the following two lemmas and is among the longest of General Topology to the author's knowledge.

LEMMA 1. *Let X be a pseudometrizable space, Y an $\aleph_0$-space, and $H \subseteq C_0(X, Y)$. If $\mathscr{U}$ is any collection of subsets of H of the form $W(K_1, \cdots, K_k; U_1, \cdots, U_k)$ with $k \in \mathbf{Z}^+$, $K_1, \cdots, K_k$ compact subsets of X and $U_1, \cdots, U_k$ open neighborhoods of 0, then there is a countable $\mathscr{U}_0 \subseteq \mathscr{U}$ such that $\bigcup_{U \in \mathscr{U}_0} U = \bigcup_{U \in \mathscr{U}} U$.*

PROOF. *Case* 1. $dc(X) \leqq 2^{\aleph_0}$. Since X is paracompact, there is a sequence $(\mathscr{V}_n)_{n=1}^{\infty}$ of locally finite open coverings of X such that, for each $n \in \mathbf{Z}^+$, $\mathscr{V}_n$ refines the collection of all open balls, in a fixed pseudometric d for X, of radius

$1/n$. Since $\mathscr{V}_n$ is locally finite and $dc(X) \leqq 2^{\aleph_0}$, $\mathrm{Card}(\mathscr{V}_n) \leqq 2^{\aleph_0}$ and hence there is an injection $\phi_n: \mathscr{V}_n \to \boldsymbol{R}\ (n \in \boldsymbol{Z}^+)$. Let $\{P_m \mid m \in \boldsymbol{N}\}$ be a countable pseudo-base for Y stable with respect to finite unions, with $P_0 = Y$. Let $\mathscr{B}$ be a countable base for the topology of $\boldsymbol{R}$. For every $n, k \in \boldsymbol{Z}^+$, every finite pairwise disjoint family $B = (B_1, \cdots, B_k) \in \mathscr{B}^k$ and every $m = (m_1, \cdots, m_k) \in \boldsymbol{N}^k$, define

$$W(n,k,m,B) = \{f \in H \mid i \in \{1, \cdots, k\},\ V \in \mathscr{V}_n \text{ and } \phi_n(V) \in B_i \Rightarrow f(V) \subseteq P_{m_i}\}.$$

Note that $W(n,k,m,B)$ may be empty. Clearly the collection $\mathscr{W}$ of all $W(n,k,m,B)$'s is countable, so that it is a subbase for a 2nd countable topology τ on H. Consequently, to prove our lemma it is enough to show that for every $f \in H$, every compact $K \subseteq X$ and every open neighborhood U of 0 such that $f \in W(K,U)$, there is a τ-open set W such that $f \in W \subseteq W(K,U)$—since this means that τ is finer than the topology having as subbase the sets $W(K,U)$ with $K \subseteq X$ compact and U open neighborhood of 0. Choose then $f \in H$, a compact $K \subseteq X$ and an open neighborhood U of 0 such that $f \in W(K,U)$. Since Y is regular, there is a closed neighborhood U' of 0 contained in U. Since f vanishes at infinity, there is a compact $K' \subseteq X$ such that $f(X \setminus K') \subseteq U'$. Since $f(X) \cup \{0\}$ is a compact subset of Y, $C = U' \cap (f(X) \cup \{0\})$ is a compact subset of Y. Hence there is P_{m_0} such that

$$f(X \setminus K') \subseteq C \subseteq P_{m_0} \subseteq U.$$

Since Y is normal (being Lindelöf), there is a closed neighborhood U'' of $f(K)$ contained in U. Since $f^{-1}(U'')$ is a neighborhood of K and K compact, there is $n \in \boldsymbol{Z}^+$ such that $\mathrm{St}(K, \mathscr{V}_n) \subseteq f^{-1}(U'')$. Since a locally finite family in a compact space is finite, the set

$$\mathscr{V}_n^* = \{V \in \mathscr{V}_n \mid V \cap (K \cup K') \neq \varnothing\}$$

is finite. Put $\{V_1, \cdots, V_k\} = \mathscr{V}_n^*$ with $V_i \neq V_j$ for $i \neq j$. Since $\phi_n(V_i) \neq \phi_n(V_j)$ for $i \neq j$, there is a pairwise disjoint family $B = (B_1, \cdots, B_k) \in \mathscr{B}^k$ such that $\phi_n(V_i) \in B_i$ for every i. Suppose $V_i \cap K \neq \varnothing$. Since $f(X) \cup \{0\}$ is compact, $\overline{f(V_i)}$ is compact. Since $f(V_i) \subseteq U''$ and U'' is closed, $\overline{f(V_i)}$ is a compact subset of U, so that there is $P_{m_i^0}$ such that

$$\overline{f(V_i)} \subseteq P_{m_i^0} \subseteq U \qquad (V_i \cap K \neq \varnothing).$$

If $V_i \cap K = \varnothing$, then put $m_i^0 = 0$, i.e. $P_{m_i^0} = P_0 = Y$. Define

$$P_{m_i} = P_{m_i^0} \cup P_{m_0} \qquad (i = 1, \cdots, k)$$

and $m = (m_1, \cdots, m_k) \in N^k$. Then clearly $W(n,k,m,B) \subseteq W(K,B)$. To prove that $f \in W(n,k,m,B)$, suppose that $\phi_n(V) \in B_i$ for $V \in \mathscr{V}_n$ and $i \in \{1, \cdots, k\}$. If $V \in \mathscr{V}_n{}^*$, then $f(V) \subseteq P_{m_i}$ by construction. If $V \notin \mathscr{V}_n{}^*$, then $V \subseteq X \setminus K'$, so that

$$f(V) \subseteq f(X \setminus K') \subseteq P_{m_0} \subseteq P_{m_i}.$$

Case 2. General case. Suppose that there is no countable $\mathscr{V} \subseteq \mathscr{U}$ such that

$$\bigcup_{V \in \mathscr{V}} V = \bigcup_{U \in \mathscr{U}} U.$$

Let Ω be the first uncountable ordinal. Since for every $\alpha < \Omega$ the set $\{\beta < \Omega \mid \beta < \alpha\}$ is countable, by our hypothesis we can construct inductively a family $(\mathscr{U}_\alpha)_{\alpha<\Omega}$ of countable subsets of $\mathscr{U}$ such that, for each α, $\mathscr{U}_\alpha \subseteq \mathscr{U}_{\alpha+1}$ and

$$\bigcup_{U \in \mathscr{U}_\beta, \beta<\alpha} U \not\supseteq \bigcup_{U \in \mathscr{U}_\alpha} U.$$

Suppose $\mathscr{U}_\alpha = \{U_n^\alpha \mid n \in \mathbf{Z}^+\}$. For every α and n, there are compact subsets $K_1^{\alpha,n}, \cdots, K_{k(\alpha,n)}^{\alpha,n}$ of X and open neighborhoods $U_1^{\alpha,n}, \cdots, U_{k(\alpha,n)}^{\alpha,n}$ of 0 such that

$$U_n^\alpha = W(K_1^{\alpha,n}, \cdots, K_{k(\alpha,n)}^{\alpha,n};\ U_1^{\alpha,n}, \cdots, U_{k(\alpha,n)}^{\alpha,n}).$$

Put $Z = \bigcup_{\alpha,n}(K_1^{\alpha,n} \times \cdots \times K_{k(\alpha,n)}^{\alpha,n})$. Clearly $dc(Z) \leqq \aleph_1 \leqq 2^{\aleph_0}$ and $\{f|_Z \mid f \in C_0(X,Y)\} \subseteq C_0(Z,Y)$. Therefore, by Case 1, the set

$$\mathscr{U}^* = \{\{f|_Z \mid f \in U_n^\alpha\} \mid \alpha < \Omega,\ n \in \mathbf{Z}^+\}$$

has a countable subset $\mathscr{V}$ such that $\bigcup_{V \in \mathscr{V}} V = \bigcup_{U \in \mathscr{U}^*} U$. To each $V \in \mathscr{V}$ there correspond $\alpha_V < \Omega$ and $n_V \in \mathbf{Z}^+$ such that $V = \{f|_Z \mid f \in U_{n_V}^{\alpha_V}\}$. Since $\mathscr{V}$ is countable, there is $\alpha < \Omega$ such that $\alpha \geqq \alpha_V$ for all $V \in \mathscr{V}$. But then

$$\bigcup_{U \in \mathscr{U}_\beta, \beta<\alpha} U \supseteq \bigcup_{U \in \mathscr{U}_{\alpha_V}, V \in \mathscr{V}} U \supseteq \bigcup_{U \in \mathscr{U}_\alpha} U,$$

a contradiction. q.e.d.

The following lemma is well known for locally compact spaces X.

LEMMA 2. *Let X be an arbitrary topological space, Y a uniform space, and H the set of all maps $X \to Y$ having a limit at infinity. If H is equipped with the topology of uniform convergence on X, then*

$$w(H) \leqq w(X) \cdot dc(Y) \cdot bc(Y)$$

provided that at least one of the later cardinals is infinite.

PROOF. Since the uniformity of uniform convergence has basis character $\leqq bc(Y)$, it is enough to prove that $dc(H) \leqq w(X) \cdot dc(Y) \cdot bc(Y)$. With this in mind, let $\mathscr{B}$ be a basis of X with cardinal equal to $w(X)$, E a dense subset of Y with cardinal equal to $dc(Y)$, and $\mathscr{W}$ a basis of the uniformity of Y made of open symmetric *entourages* with cardinal equal to $bc(Y)$. For every $n \in \mathbf{Z}^+$, every $B = (B_1, \cdots, B_n) \in \mathscr{B}^n$, every $y = (y_1, \cdots, y_{n+1}) \in E^{n+1}$, and every $W \in \mathscr{W}$, define:

(i) $f_{(B,y)}: X \to Y$ to be constantly y_i on $B_i \setminus \bigcup_{j=1}^{i-1} B_j$ $(i = 1, \cdots, n)$, and y_{n+1} outside $B_1 \cup \cdots \cup B_n$.

(ii) $g_{(B,y,W)}$ as an element of

$$H \cap \{g \mid g: X \to Y,\ (g(x), f_{(B,y)}(x)) \in W \quad (x \in X)\}$$

whenever this intersection is not empty.

Now choose $f \in H$ and $W \in \mathscr{W}$. Let y_f be a limit at infinity of f, and $W' \in \mathscr{W}$ such that $W' \circ W' \subseteq W$. There is a compact $K \subseteq X$ such that $f(X \setminus K) \subseteq W'[y_f]$. Since f is continuous and $\{W'[y] \mid y \in E\}$ is an open cover of Y, there is $B = (B_1, \cdots, B_n) \in \mathscr{B}^n$ such that $K \subseteq B_1 \cup \cdots \cup B_n$, and every B_i is contained in some $f^{-1}(W'[y])$ with $y \in E$. Define $y = (y_1, \cdots, y_{n+1}) \in E^{n+1}$ by $B_i \subseteq f^{-1}(W'[y_i])$ $(i = 1, \cdots, n)$ and $y_{n+1} \in E \cap W'[y_f]$. Then clearly

$$(f(x), f_{(B,y)}(x)) \in W \qquad (x \in X),$$

so that $g_{(B,y,W)}$ is defined and

$$(f(x), g_{(B,y,W)}(x)) \in W \circ W \qquad (x \in X).$$

Consequently the family of all maps of the form $g_{(B,y,W)}$ is dense in H. Since this family is indexed by a subset of

$$\left(\bigcup_{n=1}^{\infty} (\mathscr{B}^n \times E^{n+1}) \right) \times \mathscr{W},$$

the lemma is proved. q.e.d.

Proof of Theorem 1. Since an $\aleph_0$-space is embeddable into a $C_p(M)$ with M separable metric by [4], Y is submetrizable, i.e. its topology is finer than a metrizable topology. Let d be a metric for Y whose topology is less fine than the topology of Y. Suppose X is the topological sum of the family $(X_\gamma)_{\gamma \in \Gamma}$ of separable subspaces of X, i.e. suppose $(X_\gamma)_{\gamma \in \Gamma}$ is a Morita decomposition of X. By Lemma 2, there is a countable subset H_γ of $C_0(X_\gamma, Y)$ which is dense for the topology of uniform convergence with respect to (Y, d). Let $\tilde{X}_\gamma$ be the completion of X_γ for the weak uniformity μ_γ induced on X_γ by H_γ as set of mappings $X \to (Y, d)$. Since $\overline{f(X_\gamma)}$ is compact for all $f \in C_0(X_\gamma, Y)$, μ_γ is precompact. Since H_γ is countable,

μ_γ is pseudometrizable. Therefore $\tilde{X}_\gamma$ is a compact pseudometric space. Let $f \in C_0(X_\gamma, Y)$. Since f is uniform limit, with respect to (Y, d), of members of H_γ, $f: \mu_\gamma X_\gamma \to (Y, d)$ is uniformly continuous. But (Y, d) induces on the compact space $\overline{f(X_\gamma)}$ its unique compatible uniformity. Therefore f has a continuous extension $\tilde{f}: \tilde{X}_\gamma \to \overline{f(X_\gamma)} \subseteq Y$. Let $\tilde{X}$ be the topological sum of $(\tilde{X}_\gamma)_{\gamma \in \Gamma}$: $\tilde{X}$ is a locally compact, pseudometrizable space. To each $f \in C_0(X, Y)$ we associate the map $\tilde{f}: \tilde{X} \to Y$ which agrees on $\tilde{X}_\gamma$ with $\widetilde{f \mid_{X_\gamma}}$. In order to show that $\tilde{f} \in C_0(\tilde{X}, Y)$ choose a closed neighborhood U of 0 in Y. There is a compact $K \subseteq X$ such that $f(X \setminus K) \subseteq U$. Since the topology induced on X by $\tilde{X}$ is less fine than the topology of X because of the continuity of members of all $C_0(X_\gamma, Y)$'s, K is a compact subset of $\tilde{X}$. Hence the closure $\bar{K}$ of K in $\tilde{X}$ is again compact. Choose $x \in \tilde{X} \setminus \bar{K}$. There is a sequence $(x_n)_{n=1}^\infty$ in X converging to x in $\tilde{X}$. Since $\tilde{X} \setminus \bar{K}$ is a neighborhood of x in $\tilde{X}$, there is an n_0 such that $x_n \in \tilde{X} \setminus \bar{K}$ $(n \geqq n_0)$. Therefore $f(x_n) \in U$, and $\lim_n f(x_n) \in U$ because of the closedness of U. Then the continuity of $\tilde{f}$ implies $\tilde{f}(x) \in U$. This shows that $f \rightsquigarrow \tilde{f}$ is a map $\Phi: C_0(X, Y) \to C_0(\tilde{X}, Y)$. Obviously $\Phi(C_0(X, Y))$ is invariant under projections, and Φ is a homeomorphism into when $C_0(X, Y)$ is equipped with the topology of uniform convergence on the X_γ's, and $C_0(\tilde{X}, Y)$ with the topology τ of uniform convergence on the $\tilde{X}_\gamma$'s. Therefore we have to prove that any subset of $C_0(\tilde{X}, Y)$ which is invariant under projections is a Lindelöf space with the topology induced by τ. Fix $A \subseteq C_0(\tilde{X}, Y)$ invariant under projections. Clearly the topology of any $C_0(\tilde{X}_\gamma, Y)$ of uniform convergence on $\tilde{X}_\gamma$ is the compact-open topology. Therefore (the $\tilde{X}_\gamma$'s being 2–2 disjoint) a basis for τ is given by all sets of the form $W(K_1, \cdots, K_n; U_1, \cdots, U_n)$ with $n \in Z^+, U_1, \cdots, U_n$ open subsets of Y and $K_1, \cdots, K_n$ compact subsets of $\tilde{X}$ such that

$$(*) \qquad K_i \cap \tilde{X}_\gamma \neq \emptyset \Rightarrow K_i \subseteq \tilde{X}_\gamma \qquad (i = 1, \cdots, n).$$

Therefore it is enough to show that every cover of A made of sets of this form has a countable subcover. Let $\mathscr{U}$ be such a cover. Let $\Gamma' \subseteq \Gamma$ be countable, and $X' = \coprod_{\gamma \in \Gamma'} \tilde{X}_\gamma$ (here and below, $\coprod$ denotes topological sum). By [4, (J)], $C_0(X', Y)$ is an $\aleph_0$-space: let $\mathscr{P}$ be a countable pseudobase for $C_0(X', Y)$. For each $P \in \mathscr{P}$, let $\mathscr{U}_P$ be the set of all $U = W(K_1, \cdots, K_n; U_1, \cdots, U_n) \in \mathscr{U}$ such that $U \mid_{X'} \supseteq P$ and $0 \in U_i$ whenever $K_i \cap X' = \emptyset$. By Lemma 1, there is a countable $\mathscr{U}'_P \subseteq \mathscr{U}_P$ such that

$$\bigcup_{U \in \mathscr{U}'_P} U \big|_{\tilde{X} \setminus X'} = \bigcup_{U \in \mathscr{U}_P} U \big|_{\tilde{X} \setminus X'}.$$

Put

$$\mathscr{U}_0 = \bigcup_{P \in \mathscr{P}} \mathscr{U}_P, \qquad \mathscr{U}' = \bigcup_{P \in \mathscr{P}} \mathscr{U}'_P.$$

Let us show that $\bigcup_{U \in \mathscr{U}_0} U = \bigcup_{U \in \mathscr{U}'} U$. If $f \in \bigcup_{U \in \mathscr{U}0} U$, then there is $U \in \mathscr{U}_0$ such that $f \in U$. Since $\mathscr{P}$ is a pseudobase for $C_0(X', Y)$, there is $P \in \mathscr{P}$ such that $f|_{X'} \in P \subseteq U|_{X'}$. There is $U' = W(K_1, \cdots, K_n; U_1, \cdots, U_n) \in \mathscr{U}'_P$ such that $f|_{\tilde{X} \setminus X'} \in U'|_{\tilde{X} \setminus X'}$. Hence $f(K_i) \subseteq U_i$ whenever $K_i \cap X' = \varnothing$. If $K_i \cap X \neq \varnothing$, then $K_i \subseteq X'$ by $(*)$. Since $P \subseteq U'|_{X'}$, $f(K_i) \subseteq U_i$ whenever $K_i \cap X' \neq \varnothing$. Therefore $f \in U'$, and so $\bigcup_{U \in \mathscr{U}_0} U = \bigcup_{U \in \mathscr{U}'} U$. Now we are ready to conclude like in the original proof of [2, 2.1]. It follows from the preceding argument that we can construct inductively a sequence $(\Gamma_n)_{n=0}^{\infty}$ of countable subsets of Γ and a sequence $(\mathscr{U}_n)_{n=0}^{\infty}$ of countable subsets of $\mathscr{U}$ such that, for every $n \in N$:

(i) $\Gamma_n \subseteq \Gamma_{n+1}$.

(ii) $\bigcup_{U \in \mathscr{U}_n} U \supseteq W$ whenever $W = W(K_1, \cdots, K_n; U_1, \cdots, U_n) \in \mathscr{U}$ and $0 \in U_i$ if $K_i \cap \coprod_{\gamma \in \Gamma_n} \tilde{X}_\gamma = \varnothing$.

(iii) $W(K_1, \cdots, K_n; U_1, \cdots, U_n) \in \mathscr{U}_n \Rightarrow K_1 \cup \cdots \cup K_n \subseteq \coprod_{\gamma \in \Gamma_n} \tilde{X}_\gamma$.

Let $\Gamma_\infty = \bigcup_{n=0}^{\infty} \Gamma_n$. Choose $f \in A$. Since A is invariant under projections, A contains the map f_0 which agrees with f on $\coprod_{\gamma \in \Gamma\infty} \tilde{X}_\gamma$ and is 0 otherwise. Therefore there is $U = W(K_1, \cdots, K_n; U_1, \cdots, U_n) \in \mathscr{U}$ such that $f_0 \in U$. By (i), there is $n_0 \in N$ such that $K_i \cap \coprod_{\gamma \in \Gamma_{n0}} \tilde{X}_\gamma \neq \varnothing$ whenever $K_i \cap \coprod_{\gamma \in \Gamma\infty} \tilde{X}_\gamma \neq \varnothing$. If $K_i \cap \coprod_{\gamma \in \Gamma_{n0}} \tilde{X}_\gamma = \varnothing$, then $K_i \cap \prod_{\gamma \in \Gamma\infty} \tilde{X}_\gamma = \varnothing$, so that $0 \in U_i$. Thus (ii) implies $U \subseteq \bigcup_{U' \in \mathscr{U}_{n0}} U'$: let $U' = W(K'_1, \cdots, K'_m; U'_1, \cdots, U'_m) \in \mathscr{U}_{n_0}$ be such that $f_0 \in U'$. By (iii), $K'_1 \cup \cdots \cup K'_m \subseteq \coprod_{\gamma \in \Gamma\infty} \tilde{X}_\gamma$, so that $f(K'_i) \subseteq U'$ $(i = 1, \cdots, m)$, and $f \in U'$. Therefore $\bigcup_{n=0}^{\infty} \mathscr{U}_n$ is a countable subcover of $\mathscr{U}$.[i]

q.e.d.

Proof of Theorem 2. By [3, 8.4],

$$U_f = X \setminus f^{-1}(0)$$

is locally compact for every $f \in C_0(X, Y)$. Therefore $Z = \bigcup_{f \in C_0(X,Y)} U_f$ is an open, locally compact subspace of X, and all $f \in C_0(X, Y)$ are 0 on $X \setminus Z$. If U is an open neighborhood of 0 in Y and $f \in C_0(X, Y)$, then $f^{-1}(Y \setminus U)$ is a closed subset of X contained in a compact set as well as in Z. Thus

$$H = \{f|_Z \mid f \in C_0(X, Y)\}$$

is contained in $C_0(Z, Y)$. Let us show that H is invariant under projections. Let A be a clopen subset of Z and $f \in H$, with $f = \bar{f}|_Z$ for $\bar{f} \in C_0(X, Y)$. Let f_0 be the map $Z \to Y$ which agrees with f on A and is 0 on $Z \setminus A$. Let g be the map $X \to Y$ which agrees with f_0 on Z and is 0 on $X \setminus Z$. Since g is an extension of f_0, in order to prove that $f_0 \in H$ it is enough to show that g is continuous, since then $g \in C_0(X, Y)$. To prove the continuity of g, we have only to prove its continuity on the frontier ∂Z of Z in X. Fix $x \in \partial Z$. We have only to prove that $x_n \in Z$ and $\lim_n x_n = x$ imply $\lim_n g(x_n) = 0$. Suppose not, and argue for a contradiction. There are then a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ and an open neighborhood U of 0 in Y such that $g(x_{n_k}) \notin U$ for all k. Consequently $g(x_{n_k}) = \bar{f}(x_{n_k})$. But then $\lim_k \bar{f}(x_{n_k}) \notin U$, which is impossible since $\lim_k \bar{f}(x_{n_k}) = \bar{f}(x) = 0$. Therefore g is continuous, and H invariant under projections. Then H, equipped with the topology of pointwise convergence, is a Lindelöf space by Theorem 1. Since $f(x) = 0$ for $x \notin Z$ and $f \in C_0(X, Y)$, $C_0(X, Y)$ is a Lindelöf space for the topology of pointwise convergence. q.e.d.

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